

10.1 Let $\mathcal{M} \subset \mathbb{R}^n$ be an m dimensional submanifold of class C^k and let $\mathcal{U} \subset \mathbb{R}^n$ be an open set such that $\mathcal{M} \cap \mathcal{U} \neq \emptyset$. Show that $\mathcal{M} \cap \mathcal{U}$ is also an m dimensional submanifold of class C^k .

10.2 (a) First, let's recall some notions from linear algebra: Explain why the volume of the parallelepiped $P \subset \mathbb{R}^m$ constructed on the vectors $b_1, \dots, b_m \in \mathbb{R}^m$ satisfies

$$\text{Vol}(P) = \sqrt{\det(G)},$$

where G is the Gram matrix of b_1, \dots, b_m (i.e., the matrix whose coefficients are the inner products $g_{ij} = \langle b_i, b_j \rangle$).

(b) Let $\psi : \Omega \rightarrow \mathcal{M}$ be a C^1 local parametrization of the submanifold $\mathcal{M} \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^m$ is open and let G be the associated metric tensor. Informally explain (using the previous part) why $\sqrt{\det(G(u))} du_1 \dots du_m$ is “reasonable” as an expression for the volume of the infinitesimal coordinate parallelepiped around $\psi(u)$ on \mathcal{M} .

(c) Show that the volume of $\psi(\Omega)$, namely

$$\int_{\Omega} \sqrt{\det(G(u))} du_1 \dots du_m$$

is invariant under changes of parametrization, i.e. if $h : \Omega' \rightarrow \Omega$ is a C^1 diffeomorphism and G' is the metric tensor associated to $\psi' = \psi \circ h$, then

$$\int_{\Omega'} \sqrt{\det(G'(u'))} du'_1 \dots du'_m = \int_{\Omega} \sqrt{\det(G(u))} du_1 \dots du_m.$$

10.3 Prove that the area of a regular parametrized surface $\psi : \Omega \rightarrow S \subset \mathbb{R}^3$ of class C^1 can be computed by the formula

$$\text{Area}(S) = \iint_{\Omega} \left\| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right\| du dv.$$

10.4 Let $f : [a, b] \rightarrow \mathbb{R}$ be a C^1 function and let S be the surface of revolution in \mathbb{R}^3 obtained by rotating the graph of f around the x -axis. Prove carefully that

$$\text{Area}(S) = 2\pi \int_a^b \sqrt{1 + (f'(x))^2} |f(x)| dx.$$

10.5 (a) Give a maximal open domain on which polar coordinates define a diffeomorphism $\psi : (r, \theta) \mapsto (x, y)$.

- (b) Compute the associated metric tensor.
- (c) Deduce from it the formula for computing the area of a domain in polar coordinates.

10.6 Consider the helicoid defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x \sin(z) - y \cos(z) = 0\}.$$

- (a) Prove that the helicoid is a ruled surface (see Ex. 9.5) and describe the geometry of this surface.
- (b) Show that the map
$$\psi(u, v) = (v \cos u, v \sin u, u)$$
defines a (global) diffeomorphism between \mathbb{R}^2 and S .
- (c) .Compute the metric tensor associated with this parametrization

10.7 The catenary is the graph of the hyperbolic cosine, i.e., the curve $\alpha(t) = (t, \cosh t)$. This also happens to be the shape of a chain hanging from its two endpoints (deriving the shape of such a chain is an easy exercise in the context of a physics course).

- (a) Show that the curvature of α is given by $\kappa(t) = 1/\cosh^2(t)$.
- (b) Compute the evolute of α (this curve is called a *tractrix*).
- (c) Compute the arc length of the catenary from the initial point $\alpha(0) = (0, 1)$, then give the natural parametrization of α .
- (d) The surface of revolution of the catenary around the x -axis is called a catenoid. Compute the metric tensor of the catenoid (preferably using the natural parametrization of the catenary).

10.8 Let $\mathcal{S}_a \subset \mathbb{R}^3$ be the sphere of radius $a > 0$ centered at the origin. The stereographic projection is the map

$$\pi : \mathcal{S}_a \setminus \{(0, 0, a)\} \rightarrow \mathbb{R}^2$$

which sends a point $p = (x, y, z) \neq (0, 0, a)$ to the unique point q in the plane \mathbb{R}^2 such that the three points $(0, 0, a)$, p , and q are aligned (we view \mathbb{R}^2 as the plane $\{z = 0\}$ in \mathbb{R}^3).

Let $\psi : \mathbb{R}^2 \rightarrow \mathcal{S}_a$ denote the inverse stereographic projection.

- (a) Find an explicit formula for ψ and show that ψ is a regular parametrization of $\mathcal{S}_a \setminus \{(0, 0, a)\}$.
- (b) Compute the associated metric tensor.

- (c) Is this parametrization conformal?
- (d) Prove that stereographic projection defines a homeomorphism between the sphere and the Alexandrov compactification of \mathbb{R}^2 .

(The Alexandrov compactification of \mathbb{R}^2 is the set $\widehat{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ with the topology for which every neighborhood of a point of \mathbb{R}^2 is also a neighborhood in $\widehat{\mathbb{R}^2}$, and the complements of compact subsets of \mathbb{R}^2 form a neighborhood basis of the point ∞ .)

Remark. Sometimes one defines stereographic projection onto a plane other than the equatorial plane; in particular, one often projects onto the tangent plane at the “south pole” $(0, 0, -a)$.